MS698–3: Sediment transport processes in coastal environments
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February 11, 2003

Lecture 5: Threshold for Transport

- Force balance on particle under incipient motion.
- Critical shear stress: τ_{cr}.
- Shields Curve.

Class business

- Reading list update: revised version on-line.
- Reading for Thursday:
  Thomsen, L. and G. Gust, 2000. Sediment erosion thresholds and characteristics of resuspended aggregates on the western European continental margin, *Deep-Sea Research I*, 47: 1881-1897.
- Term paper topics due this week.
- Next week will have review for first problem set.

Materials used: Wiberg and Smith (1988); Dade et al. (1992)

Handouts

- Force balance cartoon
- Hjulstrom's Diagram: u_{cr} vs. D.
- Shield's Diagram: τ_{*cr} vs. R_*
- Diagrams from Wiberg and Smith (1988).

Definition of threshold for motion

In this lecture we look at methods developed for predicting the transition between a sediment bed being at rest, and being transported by fluid drag. Some researchers look at the problem as describing the probability distribution function that sediment within a population will move. Most often the viewpoint that transport is defined by a critical threshold is used. The threshold has been defined as a critical value of shear stress (τ_{cr} or u_{*cr}); or a critical value for velocity (u_{cr}) where below the threshold ($\tau_{cr} > \tau_b$; or $u_{cr} > u$) the flow can not move the sediment; but above the threshold it does. It is difficult to be precise about the term "transport threshold", especially in terms of laboratory or field operations. Definitions of transport include

- *first motion* or *incipient motion*: point at which only the most vulnerable particles on the bed are moved.
- *feeble motion*: the point where there is low, but consistent transport.
- *general motion*: where the entire bed is moving; or where sediment flux rates exceed some criteria.

Various data sets used different criteria for the threshold of motion. This led to a lot of scatter in seemingly comparable data sets. For the purpose of this discussion, we'll adapt Shield's criteria for transport as "feeble motion" the point at which some non-negligible portion of the sediment bed is transported.

The *Hjulstrom diagram* is commonly used to infer a critical velocity threshold for transport (see handout). Problems with this approach include the fact that it does not state explicitly which velocity to use (reference height velocity, depth-average velocity?), is valid for only water / quartz sediment mixtures, and is valid for only unidirectional flows (no waves).

We use a force balance to justify Shields curve, which relates critical shear stress (τ_{*cr}) to roughness Reynolds number (R_*). This addresses many of the problems with the Hjulstrom diagram.

Force balance on particle under incipient motion

Consider a particle that is experiencing critical force balance on the bed; that is - any increase in flow-induced drag or lift will cause the particle to be transported. Assume that the bed is flat; not sloped. There are four forces acting on this particle: drag, gravity, lift, and bed resistance, that can be quantified as follows:

- F_G = total gravitational force.

  $$F_g = (\rho_s - \rho)\, gV, \tag{1}$$

  where ρ_s is sediment density, ρ is fluid density, g is gravitational acceleration, and V is the volume of the particle. Define the *net gravitational force*, $F_N = F_g - F_L$, to take account of the lift.

- F_L = lift force caused by pressure differences around the particle due to velocity shear across the grain diameter. The lift force will only be significant very close to the bed.

  $$F_L = \rho \frac{C_L}{2} \left({u_T}^2 - {u_B}^2\right) A; \tag{2}$$

  where $C_L \approx 0.2$ is a lift coefficient, u_T and u_B are the velocities at the top and bottom of the particle, and A is the horizontal cross-sectional area.

- F_D = drag force.

  $$F_D = \frac{1}{2} \rho C_D {u_D}^2 A, \tag{3}$$

where C_D is a drag coefficient, u_D is the velocity that the flow would have if the sediment grain were absent, and A is vertical cross-sectional area.

As an example, the velocity scale u_D can be estimated by assuming a log layer near the bed; and depth-averaging the velocity profile over the range z_0 to D. This is equivalent to assuming that the flow is hydraulically rough (HRF), so that the grain extends well above the viscous sub-layer:

$$\begin{aligned} u_D &= \left[\int_{z_0}^{D} u(z)dz\right] / (D - z_0)\,; \\ &= \left[\int_{z_0}^{D} \frac{u_*}{\kappa} \ln (z/z_0) dz\right] / (D - z_0)\,; \\ &= \frac{u_*}{\kappa} (\ln (D/z_0) - 1)\,. \end{aligned} \tag{4}$$

Note: $\int \ln (z/z_0)dz = z (\ln z/z_0 - 1) + C$.

Using $\tau_b = \rho u_*^2$, this gives the drag force for hydraulically rough flows:

$$\begin{aligned} F_D &= \tau_b \frac{C_D}{2} \left[\frac{1}{\kappa} (\ln (D/z_0) - 1)\right]^2 A; \\ &= \tau_b \, \frac{C_D}{2} \, f^2 (D/z_0) \;\, A; \end{aligned} \tag{5}$$

where $f (D/z_0) = u_D/u_* = [\ln (D/z_0) - 1] /\kappa$.

- F_R is the resistance of the bed to motion, and has vertical ($(F_R)_y$) and horizontal ($(F_R)_x$) components. The vertical component of the resisting force from the constricting grain must just balance the net gravitational force ($(F_R)_y = F_N$). The horizontal component of the resisting force is therefore dependent on the net weight of the particle, and the angle of rest on the constricting particle (ϕ, the particle angle of repose);

$$\begin{aligned} (F_R)_y &= (\rho_s - \rho)\, gV - F_L; \\ (F_R)_x &= [(\rho_s - \rho)\, gV - F_L] \tan \phi. \end{aligned} \tag{6}$$

At the critical threshold for motion both the horizontal and vertical forces must balance. The vertical force balance is satisfied by the $(F_R)_y$, F_L, and F_g. The horizontal force balance is met by equating the drag and horizontal component of the resistance force $(F_D)_{cr} = (F_R)_x = (F_g - F_L) \tan \phi$. Notice that we are now using the notation for critical drag force $(F_D)_{cr}$. The ratio of drag to gravitation is

$$\frac{(F_D)_{cr}}{F_g} = \frac{\tan \phi}{1 + \frac{F_L}{F_D} \tan \phi}. \tag{7}$$

Substituting the definitions of drag and gravitational forces (equ 1 and 5) yields

$$\frac{(F_D)_{cr}}{F_g} = \frac{1}{2} \; \frac{\tau_{cr}}{(\rho_s - \rho)\, gD} \; C_D \; f^2\,(D/z_0) \; \frac{AD}{V}, \tag{8}$$

where $(F_D)_{cr}$ indicates that we are considering the critical threshold for motion; and $f\,(D/z_0) = u_D/u_* = (\ln D/z_0 - 1)\,/\kappa$. Equations 7 and 8 can be rearranged to solve for critical shear stress. Assuming spherical particles;

$$\tau_{cr} = \frac{4}{3C_D} \; (\rho_s - \rho)\, gD \; f^{-2}\,(D/z_0) \; \frac{\tan\phi}{1 + \frac{F_L}{F_D}\tan\phi}. \tag{9}$$

This gives a general equation for critical shear stress where we have assumed non-cohesive spherical particles, and not much else. Shear stresses are often non-dimensionalized to $\tau_{*cr} = \tau/\left[(\rho_s - \rho)\, gD\right]$; sometimes called a *Shield's stress*, with symbol $\theta = \tau_*$. Equation 9 can be re-written as a non-dimensional critical shear stress:

$$\tau_{*cr} = \frac{4}{3C_D} \; f^{-2}\,(D/z_0) \; \frac{\tan\phi}{1 + \frac{F_L}{F_D}\tan\phi}. \tag{10}$$

The right-hand-side of equation 10 contains three terms. (1) The drag coefficient term ($\frac{4}{3C_D}$) depends on the Reynolds number, (2) the $f^{-2}(D/z_0) = u_D/u_*$ term accounts for the shape of the velocity profile near the bed, and (3) the final term accounts for sediment angularity and shape.

Shield's diagram

Equation 10 says that the critical shear stress for transport depends on the drag coefficient, the vertical profile near the bed (as parameterized by $f^2(D/z_0)$), the ratio of lift to drag, and the angle of repose of the sediment.

The last term on the right hand side of equation 10 is nearly constant. The angle ϕ does not vary much for sediment ($\phi \approx 50^o - 70^o$ for spheres to crushed quartz). For naturally shaped sediment; $\phi \approx 60^o$; so $\tan\phi \approx 1.73$. Likewise, the ratio of lift to drag is approximately constant, $F_L/F_D \approx 0.85$.

The drag coefficient, C_D depends on the Reynolds number ($R_D = uD/\nu$), as covered in an earlier lecture. This Reynolds number can be rewritten in terms of the roughness Reynolds number ($R_* = u_* K_s/\nu$); $R_D = R_* \frac{u}{u_*} \frac{D}{K_s}$; where K_s is the physical roughness scale of the bed. Therefore, the drag coefficient C_D depends on the roughness Reynolds number, the shape of the velocity profile near the bed ($u/u_* = f(D/z_0)$), and the relative length scale of the grain D/K_s.

The non-dimensional critical shear stress therefore depends on the roughness Reynolds number, the velocity profile near the bed, and the relative length scale of the grain. The shape of the velocity profile ($u_D/u_* = f(D/z_0)$) also depends on Reynolds number. This functionality explains the success of *Shield's Curve*; a diagram relating τ_{*cr} and the roughness Reynolds number (R_*). At high values of $R_* > 100$, the curve levels off and $\tau_{*cr} = 0.05 - 0.06$. In this range, critical shear stress (τ_{cr}) increases linearly with grain diameter; and $\tau_{cr} \approx 100D$ is a good approximation if D is in cm and $\tau_c r$ in dy/cm^2.

Other researchers non-dimensionalized this relationship by using D_* or ζ_* vs. τ_{*cr} (see the Wiberg and Smith (1988) graphs). Without doing this, u_* and D are on both axis.

Mixed grain size beds

The Shields diagram assumes that the bed is well-sorted. The dependency of τ_{*cr} on D/K_s implies that different sized grains in a population may experience different transport thresholds however. This is treated in some detail in Wiberg and Smith (1988). To summarize, because the larger grains ($D/K_s > 1$) are more likely to protrude into the flow; they tend to experience lower thresholds for transport than if they were on a well-sorted bed. Likewise, the finer-grained sediments ($D/K_s < 1$) can hide within pores created by larger grains; and they require higher critical shear stresses for transport than if they were on a well-sorted bed. Therefore; they conclude that most mixed-grain sediment beds have a common shear stress for transport, defined by the critical shear stress for $K_s = D_{65}$. This is not true for *bimodal distributions*, where the finer grained portion tends to be more easily transported than the coarser grained portion.

Sloping bed

Another assumption in the previous discussion was that the sediment bed be flat. Oftentimes, however, natural sediment beds (nearshore areas, faces of dunes, etc.) have some slope to them. For a sediment bed with slope β; the geometric terms in equation 7 would be altered to account for the bed slope:

$$\frac{\cos\beta\tan\phi - \sin\beta}{1 + F_L/F_D\tan\phi}. \tag{11}$$

The result is that sediment on beds where the flow is directed down-slope have lower thresholds for critical shear stress, and vice-versa for up-hill sloping beds. For a slope of 10%; the difference in τ_{*cr} can be ± 5.

Cohesion

The force balance that we considered also assumed that the only grain-to-grain interaction was in the resisting force F_R. For fine-grained beds, a cohesive force is felt between sediment grains. Dade et al. (1992) derive a force balance for cohesive beds where they include F_A, an attractive force between sediment beds.

References

Dade, W. B., Nowell, A. R. M., and Jumars, P. A. (1992). Predicting erosion resistance of muds. *Marine Geology*, 105:285–297.

Wiberg, P. L. and Smith, J. D. (1988). Calculations of the critical shear stress for motion of uniform and heterogeneous sediments. 23(8):1471–1480.